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Absence of the holographic principle in noncommutative Chern-Simons theory

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Abstract: We examine noncommutative Chern-Simons theory on a bounded spatial domain. We argue that upon 'turning on' the noncommutativity, the edge observables, which characterized the commutative theory, move into the bulk. We show this to lowest order in the noncommutativity parameter appearing in the Moyal star product. If one includes all orders, the hamiltonian formulation of the gauge theory ceases to exist, indicating that the Moyal star product must be modified in the presence of a boundary. Alternative descriptions are matrix models. We examine one such model, obtained by a simple truncation of Chern-Simons theory on the noncommutative plane, and express its observables in terms of Wilson lines.

Keywords: Chern-Simons Theories, Non-Commutative Geometry, Boundary Quantum Field Theory.

Contents

1. Introduction

Recently Susskind [1] proposed a description for the fractional quantum Hall effect in terms of noncommutative Chern-Simons theory [2, 3, 4]. It was based on the result that the long distance behavior for a charged fluid in a strong magnetic field in the zero vortex sector can be obtained at first order in the noncommutative parameter θ. The Chern-Simons theory written on the noncommutative plane $\times$ time, like the commutative version of the theory, is an empty theory unless sources are introduced. Sources can be introduced in the commutative theory by punching holes in the plane. Alternatively, instead of a plane, one can work with a bounded spatial domain such as a disc (which is certainly appropriate for describing finite systems such as the fractional quantum Hall system). A well known result for commutative Chern-Simons theory is that all the dynamical degrees of freedom reside at the boundary [5, 6]. They are associated with the so-called 'edge states', and generate the abelian Kac-Moody algebra, providing a simple illustration of the holographic principle. In this letter we argue that the holographic principle breaks down when we go to the noncommutative theory. We claim that as noncommutativity is turned on dynamical degrees of freedom move into the bulk. This will be demonstrated at first order in θ in section 2 using canonical hamiltonian analysis.

In the remaining sections we attempt to formulate the theory at higher order. The central problem of course is how to define an 'edge' in noncommutative theory. In section 3 we show that if the noncommutative Chern-Simons theory is formulated in terms of the Moyal star product it ceases to be a well defined gauge theory in the presence of a boundary. Although the theory makes sense at every order of θ, the class of gauge transformations becomes more and more restricted at higher and higher orders. The conclusion is that the Moyal star product must be modified in the

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presence of a boundary for a consistent theory. Alternatively, one can try to formulate the theory in terms of finite dimensional matrix models. This was the approach followed in [7], and it will also be examined in section 4. Our model is obtained by a simple truncation of Chern-Simons theory on the noncommutative plane, and except for the absence of a potential energy term which does not significantly affect the canonical formalism, is equivalent to that in [7]. As shown in [7], observables can be expressed in terms of Wilson lines [8]. One challenge for matrix models, which we are currently pursuing, is to demonstrate how the continuous manifold with boundary is recovered in the commutative limit. A successful resolution should also give insight into the correct modification of the Moyal star product.

We comment that there are two equivalent approaches to studying the commutative Chern-Simons theory on a disc which may be pursued in the noncommutative theory. One is to restore the gauge invariance broken at the boundary by introducing, by hand, edge variables and then eliminating the gauge degrees of freedom with a judicious gauge choice. The other, which requires no additional degrees of freedom or a gauge choice, is to do a canonical analysis of the original system. This straightforward approach was used in [6], and is easily adapted here to the noncommutative theory. Upon expanding the Moyal star to all orders we show the impossibility of implementing gauge transformations in the canonical theory. On the other hand, applying the former approach to noncommutative Chern-Simons theory appears to be subtle. One can try to introduce a boundary term depending on new variables, to the classical action for the purpose of restoring the gauge invariance. However, as seen in [3], it depends on an infinite number of derivatives (normal to the surface) and is therefore technically not a boundary term. The gauge degrees in the bulk are then entangled with the new degrees of freedom, and it would be remarkable to find a gauge where all the physical degrees of freedom end up at the edge. A systematic canonical treatment for such a nonlocal system does not look easy, but we expect that a careful treatment should lead to similar results to what we find.

2. First order noncommutativity

Susskind's description of a nonlinear fluid in a strong magnetic field [1] can be expressed in terms of potentials $A_\mu(x)$, $\mu = 0, 1, 2$, where $A_i(x)$, $i = 1, 2$, relate the coordinates x^i of a co-moving frame (with uniform density) to space frame coordinates X^i via

$$X^i = x^i + \theta\epsilon^{ij} A_j(x)\,. \tag{2.1}$$

θ is associated with the inverse density of the fluid, and is identified with the noncommutativity parameter. The time component $A_0(x)$ plays the role of a Lagrange multiplier, restricting the fluid to the zero vortex sector. The action is

$$S = \frac{k}{4\pi}\int_{M^3} d^3x\, \epsilon^{\mu\nu\lambda}\left(\partial_\mu A_\nu A_\lambda + \frac{\theta}{3}\{A_\mu, A_\nu\}A_\lambda\right), \qquad \mu,\nu,\ldots = 0,1,2\,. \tag{2.2}$$

The derivatives are with respect to x^μ, x^0 being time and $\epsilon^{\mu\nu\lambda}$ is totally antisymmetric with $\epsilon^{012} = 1$. The first term in parentheses is the abelian Chern-Simons term, while the second gives the first order noncommutative correction. The bracket is defined by

$$\{A, B\} = \epsilon^{ij}\partial_i A \partial_j B\,, \quad i,j,\ldots = 1,2\,, \tag{2.3}$$

$\epsilon^{ij} = \epsilon^{0ij}$. When the three-dimensional space-time manifold M^3 is $\mathbb{R}^3$ the action is invariant with respect to infinitesimal gauge transformations

$$\delta A_\mu = \partial_\mu\lambda + \theta\{A_\mu, \lambda\}\,, \tag{2.4}$$

up to order θ. For $\theta = 0$, the action (2.2) also has a general diffeomorphism symmetry. The term linear in θ breaks this symmetry to the group of diffeomorphisms with unit jacobian on the two dimensional time-slice, the so-called 'area preserving diffeomorphisms'. Infinitesimal diffeomorphisms are implemented via a Lie derivative $\mathcal{L}_\xi$. To restrict to area preserving diffeomorphisms one takes vectors ξ with components $\xi^i = \epsilon^{ij}\partial_j\eta$. Acting on a one form A

$$\begin{aligned}\mathcal{L}_\xi A &= i_\xi dA + d i_\xi A\\ &= i_\xi\left(dA + \frac{\theta}{2}\{A, A\}\right) + d i_\xi A + \theta\{A, i_\xi A\}\,,\end{aligned} \tag{2.5}$$

where A is the one form $A = A_i dx^i$, d is the exterior derivative and i denotes contraction. The first term on the last line vanishes at the level of the equations of motion, while the latter two define a gauge variation (2.4) with $\lambda = i_\xi A$. The two transformations (2.5) and (2.4) are then not independent after imposing the equations of motion. As a result, only one set of symmetry generators appears in the hamiltonian formulation of this theory.

As in the commutative case, gauge invariance is broken if the space-time domain M^3 is a bounded region. In that case gauge variations (2.4) of the action lead to

$$\delta S = \frac{k}{4\pi}\int_{M^3} d^3x \epsilon^{\mu\nu\lambda}\left(\partial_\mu\lambda\partial_\nu A_\lambda + \theta\Big[\{A_\mu\partial_\nu A_\lambda, \lambda\} + \frac{1}{3}\{\partial_\mu\lambda A_\nu, A_\lambda\}\Big]\right) + \mathcal{O}(\theta^2)\,, \tag{2.6}$$

which using Stoke's law can be written on the boundary of M^3. We consider in particular M^3 being a two dimensional disc $\times$ time. If all the field degrees of freedom are unrestricted, then the boundary terms vanish only after requiring λ and its first derivatives to vanish at the boundary. (We recall that in commutative Chern-Simons theory ($\theta = 0$) with a boundary, it is sufficient to have gauge transformations vanish at the boundary for gauge invariance, and there is no need to impose conditions on its derivatives [6]).

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In the canonical formulation of the theory the Poisson structure is determined by the first term in parentheses in (2.2). Hence the Poisson structure for the non-commutative Chern-Simons theory is equivalent to that of the commutative theory. The equal-time Poisson brackets are

$$\{A_i(x), A_j(y)\}_{PB} = \frac{2\pi}{k}\epsilon_{ij}\delta^2(x-y)\,, \tag{2.7}$$

where the field degrees of freedom $A_i(x)$, $i = 1, 2$, are subject to the Gauss law constraint

$$\partial_1 A_2 - \partial_2 A_1 + \theta\{A_1, A_2\} \approx 0\,. \tag{2.8}$$

Following [6], for the constraint to have meaning when the spatial domain has a boundary one should introduce a distribution function Λ and write it instead according to

$$G[\Lambda] = \frac{k}{2\pi}\int_{\bar{M}^2} d^2x\,\Lambda\,(\partial_1 A_2 - \partial_2 A_1 + \theta\{A_1, A_2\}) \approx 0\,, \tag{2.9}$$

where $\bar{M}^2$ is a time-slice of M^3. In order for $G[\Lambda]$ to be differentiable with respect to $A_i(x)$ we need to impose conditions on Λ at the boundary $\partial\bar{M}^2$. As in the commutative theory, this implies Λ vanishes on $\partial\bar{M}^2$. For such distributions

$$\frac{\delta G[\Lambda]}{\delta A_i(x)} = \frac{k}{2\pi}\epsilon^{ij}D_j\Lambda(x)\,, \tag{2.10}$$

where $D_j\Lambda = \partial_j\Lambda + \theta\{A_j, \Lambda\}$. We then get the following algebra of constraints

$$\{G[\Lambda], G[\Lambda']\}_{PB} = \frac{k\theta}{2\pi}\int_{\bar{M}^2} (d\Lambda\{A, \Lambda'\} - d\Lambda'\{A, \Lambda\} + \theta\,\{A, \Lambda\}\{A, \Lambda'\})\,, \tag{2.11}$$

with Λ', like Λ, vanishing on $\partial\bar{M}^2$. This algebra closes (including θ^2 terms) provided we also require that the first derivatives of Λ and Λ' vanish on $\partial\bar{M}^2$. The distribution functions Λ thus satisfy the same boundary conditions as the gauge transformations λ needed to make the gauge variations of S (2.6) vanish. Then

$$\{G[\Lambda], G[\Lambda']\}_{PB} = G[\theta\{\Lambda, \Lambda'\}]\,, \tag{2.12}$$

and $G[\Lambda]$ are the first class constraints which generate gauge transformations:

$$\delta A_i(x) = \{G[\Lambda], A_i(x)\}_{PB} = D_i\Lambda(x)\,. \tag{2.13}$$

Observables of this theory should be gauge invariant. This is the case, up to order θ, for the following class of variables

$$q(\Xi) = \frac{k}{2\pi}\int_{\bar{M}^2} d^2x\;\epsilon^{ij}\left(\partial_i\Xi + \frac{\theta}{2}\epsilon^{k\ell}\partial_k\partial_i\Xi\,A_\ell\right)A_j\,, \tag{2.14}$$

where Ξ are distributions. Unlike with Λ in $G[\Lambda]$, no boundary conditions need to be imposed on Ξ for $q(\Xi)$ to be differentiable in $A_i(x)$. In the commutative limit

$\theta \to 0$, (2.14) are edge variables as $q(\Xi)$ and $q(\Xi')$ for two distributions Ξ and Ξ' with the same boundary values are weakly equivalent (i.e., they only differ by a Gauss law constraint) [6]. This is however not the case for nonzero θ. Up to order θ

$$q(\Xi) - q(\Xi') = -G[\Delta] + \frac{k\theta}{4\pi}\int_{\bar{M}^2} d^2x\, \epsilon^{ij}\epsilon^{k\ell}\left(\partial_i\Delta\,(\partial_j A_k - \partial_k A_j) - \partial_k\Delta\,\partial_i A_j\right) A_\ell\,, \tag{2.15}$$

where $\Delta = \Xi - \Xi'$, with Ξ and Ξ' having the same boundary values. In writing the right hand side we also assumed that the first derivatives of Ξ and Ξ' have the same boundary values. Although $\partial_j A_k - \partial_k A_j$ vanishes at zeroth order by the equations of motion, the integral in (2.15) is not (weakly) zero at this order because it is not proportional to the Gauss law. This is since the distribution involves $\partial_i\Delta$, which does not have the appropriate boundary conditions.[1] Thus $q(\Xi)$ and $q(\Xi')$ are not in general equivalent and consequently such variables are bulk dependent. In the commutative limit the Poisson bracket algebra of these observables gives the abelian Kac-Moody algebra written on the boundary of $\bar{M}^2$. When $\theta \neq 0$ we get an algebra which can no longer be written on the boundary. Up to order θ,

$$\begin{aligned}\{q(\Xi), q(\Xi')\}_{PB} &= \frac{k}{2\pi}\int_{\bar{M}^2}\left(d\Xi\, d\Xi' + \theta\left(\{d\Xi, \Xi'\} - \{d\Xi', \Xi\}\right) A\right)\\ &= \frac{k}{2\pi}\int_{\partial\bar{M}^2}\Xi d\Xi' + \frac{k\theta}{2\pi}\int_{\partial\bar{M}^2}\{\Xi,\Xi'\}A - \frac{k\theta}{2\pi}\int_{\bar{M}^2}\{\Xi,\Xi'\}dA\,. \end{aligned}\tag{2.16}$$

The first two terms are expressed on the boundary, the first of which leads to the usual abelian Kac-Moody algebra. Its Fourier decomposition along the one dimensional boundary gives the usual form for the algebra. On the other hand, the last term necessarily involves the bulk. It does not vanish at this order since the distribution $\{\Xi, \Xi'\}$ does not satisfy the appropriate boundary conditions for this term to be proportional to the Gauss law.

Another set of nonlocal gauge invariant observables are the generalization of Wilson loops. Up to order θ they take the form

$$W(C) = \int_C \left(A + \theta\epsilon^{ij}A_i\left(\partial_j A - \frac{1}{2}dA_j\right)\right). \tag{2.17}$$

C is an arbitrary closed curve in $\bar{M}^2$. Using Stoke's law and repeated applications of the Gauss law they (weakly) vanish up to order θ when the interior C_{int} of C is topologically trivial. More generally, they are path independent, in agreement with the commutative case.

[1]To obtain the Gauss law, and consequently $q(\Xi) \approx q(\Xi')$, we need to require further that the second derivatives of Ξ and Ξ' agree on the boundary.

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3. Higher order noncommutativity

It was remarked that although first order Chern-Simons theory describes the long range effects of quantum Hall systems it does not describe discrete electron effects [1]. For the latter, Susskind suggests to study full blown noncommutative theory. One proposal is the Chern-Simons theory written on the noncommutative plane, which can also be expressed on a three-dimensional manifold using the Moyal star product. The action is [1, 2, 3, 4]

$$S = \frac{k}{4\pi} \int_{M^3} d^3x \, \epsilon^{\mu\nu\lambda} \left(\partial_\mu A_\nu \star A_\lambda + \frac{2}{3} A_\mu \star A_\nu \star A_\lambda \right) . \tag{3.1}$$

The $\star$ denoting the Moyal star product on a time-slice of M^3 is defined by

$$\star = \exp \left(\frac{\theta}{2} \epsilon_{ij} \overleftarrow{\partial_i} \overrightarrow{\partial_j} \right) , \qquad i, j = 1, 2 . \tag{3.2}$$

It has been argued that the level k is an integer and this gives quantized filling fractions [4]. From (3.1) one easily recovers the action (2.2) for the case $M^3 = \mathbb{R}^3$ by truncating the theory to first order in the noncommutativity parameter θ, since the first order term in the Moyal star product is proportional to the bracket (2.3). Furthermore, higher orders preserve the area preserving diffeomorphism symmetry. When $M^3 = \mathbb{R}^3$, (3.1) can be written as

$$\frac{k}{4\pi} \int_{M^3} d^3x \, \epsilon^{ij} \left(\partial_0 A_i A_j + 2A_0 (\partial_i A_j + A_i \star A_j) \right) , \tag{3.3}$$

the first term in the integrand giving the usual Poisson structure, while the second gives the new Gauss law.

For the case of M^3 with a spatial boundary, it has been claimed that (3.3) is equivalent to a noncommutative Wess-Zumino-Witten model written on the boundary [3]. However, the quantization of such a theory has an obstruction, which is evident in the hamiltonian formulation. The Gauss law constraint now takes the form

$$G[\Lambda] = \frac{k}{2\pi} \int_{\bar{M}^2} d^2x \Lambda(x) \epsilon_{ij} (\partial_i A_j + A_i \star A_j) \approx 0 , \tag{3.4}$$

and for differentiability one must impose that *all* spatial derivatives of Λ vanishes on the boundary of $\bar{M}^2$. Then the only allowable distributions Λ are non-analytic functions. It is therefore impossible to implement (analytic) gauge transformations in the canonical formulation of the theory. Note that this conclusion is unaltered if we replace the pointwise product of the distribution Λ in (3.4) by a star product.

On the other hand, a gauge theory based on the Moyal star product can be defined at each order in the noncommutative parameter, but as the order becomes higher the gauge symmetries become more constrained. At second order, the system is identical to the first because the Moyal star commutator only contributes at odd order. In that case one only needs to add the term

$$\frac{k\theta^2}{12\pi}\int_{\bar{M}^2} d^2x\,\epsilon^{ij}\epsilon^{k\ell}\epsilon^{mn}\,\partial_i\partial_k\partial_m\Xi\,A_jA_\ell A_n \tag{3.5}$$

to (2.14) to find gauge invariant observables valid to order θ^2.[2] On the other hand, at third order the Gauss law gets modified to

$$G[\Lambda]=\frac{k}{2\pi}\int_{\bar{M}^2} d^2x\Lambda\,\epsilon_{ij}\left(\partial_iA_j+\frac{\theta}{2}\{A_i,A_j\}+\frac{1}{6}\left(\frac{\theta}{2}\right)^3\{\{\{A_i,A_j\}\}\}\right)\approx 0\,. \tag{3.6}$$

Differentiability now demands that Λ and all its derivatives up to second order vanish on the boundary. Additional conditions result from demanding that the Poisson algebra of Gauss law closes. One thus obtains restrictions on the gauge theory which were not present at first order. Further restrictions occur at each odd order.

4. Matrix model

The difficulties encountered above strongly indicate that the Moyal star product must be modified in the presence of a boundary. An alternative way to proceed is to write down a finite dimensional matrix version of the theory. This was done in [7]. Here we perform the canonical analysis for a similar model, obtained by a simple truncation of the Chern-Simons theory on the noncommutative plane.

Consider the underlying space-time to be $\mathcal{M}^2_F\times\mathbb{R}$, where $\mathcal{M}^2_F$ is a matrix algebra generated by some noncommuting coordinates $\mathbf{a}$ and $\mathbf{a}^\dagger$ written in some irreducible matrix representation, and $\mathbb{R}$ corresponds to the time. The analogue of potentials are matrices which we denote by $\hat{A}_\mu$, $\mu=0,+,-$. ($\hat{A}_\pm$ are taken to be complex with $\hat{A}_\pm^\dagger=\hat{A}_\mp$.) They are functions on $\mathcal{M}^2_F\times\mathbb{R}$, and the standard Chern-Simons lagrangian is

$$L=\frac{k}{4\pi}\,\epsilon^{\mu\nu\lambda}\,\mathrm{Tr}\left(\partial_\mu\hat{A}_\nu\hat{A}_\lambda+\frac{2}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda\right). \tag{4.1}$$

∂_0 is just an ordinary time derivative which we also denote with a dot. We define spatial derivatives ∂_i of some matrix $\hat{B}$ by

$$\partial_+\hat{B}=[\hat{B},\mathbf{a}^\dagger]\,,\qquad \partial_-\hat{B}=-[\hat{B},\mathbf{a}]\,. \tag{4.2}$$

When $\mathbf{a}^\dagger$ and $\mathbf{a}$ are the standard creation and annihilation operators of a harmonic oscillator, $\mathcal{M}^2_F$ corresponds to the noncommuting plane. In that case, the matrix

[2] The sum can be written as a truncation of the following suggestive formula

$$\frac{k}{2\pi\theta}\int_{\bar{M}^2} d^2x\exp\left(\theta\epsilon^{ij}A_j(x)\frac{\partial}{\partial y^i}\right)\Xi(y)|_{y=x}\,.$$

It remains to be shown whether this formula is valid beyond second order.

representations are infinite dimensional and the action $\int dx^0 L$ is equivalent to (3.1) with domain $M^3 = \mathbb{R}^3$. The theory is invariant with respect to gauge variations, which take the form

$$\delta \hat{A}_\mu = \partial_\mu \hat{\lambda} + [\hat{A}_\mu, \hat{\lambda}]\,. \tag{4.3}$$

$\hat{\lambda}$ being a matrix with infinitesimal elements. An explicit calculation shows

$$\delta L = \frac{k}{2\pi}\,\epsilon^{ij}\,\mathrm{Tr}\,\hat{A}_0 \partial_i \partial_j \hat{\lambda} = \frac{k}{2\pi}\,\mathrm{Tr}[\mathbf{a}, \mathbf{a}^\dagger][\hat{\lambda}, \hat{A}_0]\,, \tag{4.4}$$

($\epsilon^{+-} = 1$), up to total time derivatives. For the noncommutative plane, $\mathbf{a}$ and $\mathbf{a}^\dagger$ satisfy

$$[\mathbf{a}, \mathbf{a}^\dagger] \propto \mathbb{1}\,, \tag{4.5}$$

and as a result L is gauge invariant.

Eq. (4.5) is a necessary condition for gauge invariance, and since it only has infinite dimensional solutions, it follows that finite matrix models break gauge invariance with respect to the most general variations of the form (4.3). This is analogous to the breaking of gauge invariance by bounded domains in continuum theories. Now consider truncating the harmonic oscillator Hilbert space to get a finite system. We can write

$$[\mathbf{a}]_{\beta\alpha} = \sqrt{\alpha}\,\delta_{\beta,\alpha-1}\,, \qquad [\mathbf{a}^\dagger]_{\beta\alpha} = \sqrt{\alpha+1}\,\delta_{\beta,\alpha+1}\,, \qquad \alpha, \beta = 0, 1, \ldots, N\,. \tag{4.6}$$

Then $\hat{A}_\mu$ are $(N+1)\times(N+1)$ matrices and

$$[\mathbf{a}, \mathbf{a}^\dagger] = \mathbb{1} - (N+1)\,\mathbb{P}\,, \tag{4.7}$$

where $\mathbb{P}$ is a projector which projects out the N^{th} level. From (4.4) gauge invariance for arbitrary variations (4.3) is violated by the highest level. The highest level thus plays a role similar to the edge in continuum Chern-Simons theory. The action $\int dx^0 L$ is invariant only upon restricting the matrix $\hat{\lambda}$ to the form

$$\begin{pmatrix} \bar{\lambda} & 0 \\ 0 & \lambda_{NN} \end{pmatrix}, \tag{4.8}$$

where $\bar{\lambda}$ is an $N \times N$ matrix. Upon performing a unitary transformation we can recover the essential features of [7]. By transforming $\mathbf{a} \to U\mathbf{a}U^\dagger$ and $\mathbf{a}^\dagger \to U\mathbf{a}^\dagger U^\dagger$, for unitary matrix U, (4.7) goes to the Gauss law of [7], with the appearance of gauge degrees of freedom associated with the last column of U.

Once again we examine the canonical formalism. We rewrite (4.1) as

$$L = -\frac{k}{4\pi}\,\epsilon^{ij}\,\mathrm{Tr}\,\hat{A}_i \dot{\hat{A}}_j + \frac{k}{2\pi}\,\epsilon^{ij}\,\mathrm{Tr}\,\hat{A}_0(\partial_i \hat{A}_j + \hat{A}_i \hat{A}_j)\,. \tag{4.9}$$

The first term in the trace in (4.9) defines the Poisson structure,

$$\{(\hat{A}_i)_{\alpha\beta}, (\hat{A}_j)_{\gamma\delta}\}_{PB} = \frac{2\pi}{k}\epsilon_{ij}\delta_{\alpha\delta}\delta_{\beta\gamma}\,, \tag{4.10}$$

while the second gives the Gauss law, which can be expressed in terms of $\hat{B}_+ = \hat{A}_+ - \mathbf{a}^\dagger$ and $\hat{B}_- = \hat{A}_- + \mathbf{a}$ according to

$$\hat{G} = [\hat{B}_+, \hat{B}_-] - [\mathbf{a}, \mathbf{a}^\dagger] \approx 0 \,. \tag{4.11}$$

The algebra of constraints is given by

$$\begin{aligned} \{\hat{G}_{\alpha\beta}, \hat{G}_{\gamma\delta}\}_{PB} &= \frac{2\pi}{k}\Big([\hat{B}_+, \hat{B}_-]_{\gamma\beta}\delta_{\alpha\delta} - [\hat{B}_+, \hat{B}_-]_{\alpha\delta}\delta_{\gamma\beta}\Big) \\ &= \frac{2\pi}{k}\Big(\hat{G}_{\gamma\beta}\delta_{\alpha\delta} - \hat{G}_{\alpha\delta}\delta_{\gamma\beta} + (N+1)(\delta_{\alpha N}\delta_{\delta N}\delta_{\beta\gamma} - \delta_{\gamma N}\delta_{\beta N}\delta_{\alpha\delta})\Big) \,, \end{aligned} \tag{4.12}$$

where we used (4.7). From (4.12) there are then N^2+1 first class constraints $\hat{G}_{\bar{\alpha}\bar{\beta}}$, $\bar{\alpha}, \bar{\beta}, \ldots = 0, 1, \ldots, N-1$, and $\hat{G}_{NN}$, along with $2N$ second class constraints $\hat{G}_{\bar{\alpha}N}$ and $\hat{G}_{N\bar{\alpha}}$. This means that there are a total of $2N$ independent gauge invariant quantities in the matrix elements of $\hat{A}_+$ and $\hat{A}_-$. To construct them we can use the transformation properties of $\hat{B}_i$

$$\delta(\hat{B}_i)_{\alpha\beta} = \{(\hat{B}_i)_{\alpha\beta}, \mathrm{Tr}\,\hat{\lambda}\hat{G}\}_{PB} = -\frac{2\pi}{k}[\hat{B}_i, \hat{\lambda}]_{\alpha\beta} \,. \tag{4.13}$$

For this to be a gauge variation $\hat{\lambda}$ must have the form (4.8). So for example, Tr $\hat{B}_i$ and $(\hat{B}_i)_{NN}$ are gauge invariant. $(\hat{B}_i)_{\bar{\alpha}N}$ and $(\hat{B}_i)_{N\bar{\alpha}}$ gauge transform as components of a vector and transpose vector with respectively to variations $\bar{\lambda}$ (and have opposite charges associated with variations λ_{NN}), from which we get four quadratic invariants $q_{ij} = \sum_{\bar{\alpha}=0}^{N-1}(\hat{B}_i)_{N\bar{\alpha}}(\hat{B}_j)_{\bar{\alpha}N}$. More invariants are obtained with $(\hat{B}_i)_{\bar{\alpha}\bar{\beta}}$, which transforms under the adjoint action. For example, we can write something analogous to Wilson lines [8]

$$W(\mu) = \overline{\mathrm{Tr}} e^{\mu_k \hat{B}_k} \qquad \text{and} \qquad W_{ij}(\mu) = \sum_{\bar{\alpha},\bar{\beta}=0}^{N-1} (\hat{B}_i)_{N\bar{\alpha}}\,(e^{\mu_k \hat{B}_k})_{\bar{\alpha}\bar{\beta}}\,(\hat{B}_j)_{\bar{\beta}N} \,, \tag{4.14}$$

where $\overline{\mathrm{Tr}}$ indicates a trace over matrix indices running from 0 to $N-1$. In fact, all the independent gauge invariant degrees of freedom except $(\hat{B}_i)_{NN}$ can be written in this form. To compute the algebra of the observables we should apply the Dirac brackets resulting from the second class constraints:

$$\begin{aligned} \{\mathcal{A}, \mathcal{B}\}_{DB} = \{\mathcal{A}, \mathcal{B}\}_{PB} - \frac{k}{2\pi(N+1)}\Big(&\{\mathcal{A}, \hat{G}_{\bar{\alpha}N}\}_{PB}\{\hat{G}_{N\bar{\alpha}}, \mathcal{B}\}_{PB} - \\ &- \{\mathcal{A}, \hat{G}_{N\bar{\alpha}}\}_{PB}\{\hat{G}_{\bar{\alpha}N}, \mathcal{B}\}_{PB}\Big) \,. \end{aligned} \tag{4.15}$$

Alternatively, it is possible to avoid dealing with the second class constraints and the resulting Dirac brackets by replacing the above system by a slightly simpler one where the **only** constraints that appear are the first class ones $\hat{G}_{\bar{\alpha}\bar{\beta}}$ and $\hat{G}_{NN}$.[3] For this we can modify the Chern-Simons lagrangian so that matrix elements $(\hat{A}_0)_{\bar{\alpha}N}$ and $(\hat{A}_0)_{N\bar{\alpha}}$ of the Lagrange multiplier $\hat{A}_0$ in (4.9) are absent. Since we should then also restrict gauge variations of $\hat{A}_0$ to be of the form (4.8), L is again gauge invariant for the restricted transformations. The previous $2N$ second class constraints then are absent, and there are additional $2N$ independent gauge invariant quantities in the matrix elements of $\hat{A}_+$ and $\hat{A}_-$ making up the phase space. These degrees of freedom are once again spanned by (4.14).

Finally we mention a few words about limits. Although it is a simple matter to recover Chern-Simons theory on the noncommutative plane (just take $N \to \infty$), the same cannot be said for recovering the continuum Chern-Simons theory on a bounded spatial manifold. With this in mind we introduce another parameter, call it $\hbar$, in the matrix theory by rescaling $\mathbf{a}$ and $\mathbf{a}^\dagger$ by $\sqrt{\hbar}$. We expect the appropriate limit is then $N \to \infty$ and $\hbar \to 0$ with $N\hbar = \text{const}$, but a rigorous proof of this is still missing. In particular, it is not apparent how to obtain edge variables from (4.14) in the limit.

A possible approach for recovering the spatial coordinates used in the previous sections is to apply the modified coherent states of [9]. Here we can try truncating the usual coherent states

$$|\zeta\rangle = \frac{1}{\sqrt{\mathcal{N}(|\zeta|^2)}} \sum_{\alpha=0}^{N} \frac{1}{\alpha!} \left(\frac{\zeta}{\hbar}\right)^{\alpha} (\mathbf{a}^\dagger)^\alpha |0\rangle \,. \tag{4.16}$$

The normalization condition $\langle\zeta|\zeta\rangle = 1$ fixes $\mathcal{N}(|\zeta|^2)$:

$$\mathcal{N}(|\zeta|^2) = \sum_{\alpha=0}^{N} \frac{1}{\alpha!} \left(\frac{|\zeta|^2}{\hbar}\right)^{\alpha} \equiv \mathrm{e}_N \left(\frac{|\zeta|^2}{\hbar}\right) . \tag{4.17}$$

This 'coherent state' is almost (up to the last state) an eigenstate for $\mathbf{a}$

$$\mathbf{a}|\zeta\rangle = \zeta|\zeta\rangle - \frac{\zeta^{N+1}}{\sqrt{N!\,\hbar^N\, \mathrm{e}_N(|\zeta|^2/\hbar)}} |N\rangle \,. \tag{4.18}$$

Now it is easy to see how the noncommutative plane is attained. This limit corresponds to the case when $N \to \infty$ while $\hbar$ is kept fixed. Then the norm of the last term in eq. (4.18) becomes much less then ζ for all $|\zeta|^2 \ll N\hbar$. So in this limit we recover the standard coherent states for $\mathbb{R}^2$. Furthermore, using techniques of [9] we can construct the Voros star product, which is equivalent to the Moyal star product [9, 10].

[3] This appears closer to the continuum Chern-Simons theory since the latter has no analogue of the second class constraints $\hat{G}_{\bar{\alpha}N}$ and $\hat{G}_{N\bar{\alpha}}$.

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The other limit $N \to \infty$ and $\hbar \to 0$ with $N\hbar = \text{const}$ is more complicated because it demands a more accurate definition of the coherent states. Following [9], the integration measure $d\mu(\zeta, \bar{\zeta})$ can be easily computed. Writing

$$d\mu(\zeta, \bar{\zeta}) = \frac{i}{2\pi\hbar}\Theta_N\left(\frac{|\zeta|^2}{\hbar}\right) d\zeta \wedge d\bar{\zeta}\,,$$

from completeness Θ_N should satisfy

$$\int_0^\infty \frac{d\rho\, \rho^{2n+1}\, \Theta_N(\rho^2)}{e_N(\rho^2)} = \frac{n!}{2}\,, \qquad \text{for } n = 0, 1, 2, \ldots, N\,. \tag{4.19}$$

A solution is

$$\Theta_N(\rho^2) = \frac{\Gamma[N+1, \rho^2]}{\Gamma[N+1]} = e^{-\rho^2} e_N(\rho^2)\,,$$

where $\Gamma[N, x]$ denotes the incomplete gamma function. In the limit $N \to \infty$ and $\hbar \to 1/N$,

$$\Theta_N(|\zeta|^2/\hbar) \to \begin{cases} 1\,, & |\zeta| < 1 \\ 0\,, & |\zeta| > 1\,. \end{cases}$$

Although this is suggestive of the measure on a disc of unit radius, more remains to be shown to argue that one recovers a disc. For example, one must show that the star product reduces to a pointwise product and that gauge invariant observables move to the 'edge'. Whether this happens depends on the correctness of the above limit, and as well as the correctness of the matrix model.

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